

On rack colorings for surface-knot diagrams without branch points

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Abstract

Racks do not give us invariants of surface-knots in general. For example, if a surface-knot diagram has branch points (and a rack which we use satisfies some mild condition), then it admits no rack colorings. In this paper, we investigate rack colorings for surface-knot diagrams without branch points and prove that rack colorings are invariants of S^2 -knots. We also prove that rack colorings for S^2 -knots can be interpreted in terms of quandles, and discuss a relationship with regular-equivalences of surface-knot diagrams.

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1. Introduction

A *quandle* [11, 12] is an algebraic system with three axioms which correspond to the Reidemeister moves, and is useful for studying classical knots in the 3-space. For example, for a given quandle, we have an invariant of oriented knots, called a *quandle coloring*, which has been extensively studied by many researchers (cf. [5, 8, 9]). There is an algebraic system, called a *rack* [6], similar to a quandle. It has two axioms which correspond to the framed Reidemeister moves, and is useful for studying framed classical knots in the 3-space. We note that a quandle is a rack by definition. For example, for a given rack, we have an invariant of oriented framed knots, called a *rack coloring*. Rack colorings themselves are not invariants of (unframed) oriented knots. However, by using rack colorings, Nelson [13] constructed an invariant of oriented knots. Later, the

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second author and Taniguchi [19] gave interpretation of his invariant in terms of quandles.

Quandles are also useful for studying surface-knots in the 4-space. For example, for a given quandle, we also have an invariant of surface-knots, called a *quandle coloring*. A quandle coloring for surface-knots has also been extensively studied by many researchers (cf. [10, 15, 18]). On the contrary, as far as the authors know, there are no study of surface-knots by using rack theory at present. (We note that framed surface-knots, more generally framed submanifolds of the n -space of codimension 2 with $n \geq 3$, were studied in [7] by using rack theory.) One of reasons is that surface-knot diagrams may have branch points. For example, for a given rack, we can also define a *rack coloring* for surface-knot diagrams in a way similar to quandle colorings. However, if a surface-knot diagram has branch points (and a rack which we use satisfies some mild condition), then it admits no rack coloring. We also observe in Example 3.1 that even if surface-knot diagrams which we consider have no branch points, rack colorings are not invariants of surface-knots in general.

In this paper, we investigate rack colorings for surface-knot diagrams without branch points and prove that rack colorings are invariants of S^2 -knots (Theorem 3.2). We also prove that rack colorings for S^2 -knot diagrams without branch points can be interpreted in terms of quandles (Theorem 3.3). We note that Theorem 3.3 implies Theorem 3.2. In the final section, we also discuss a relationship with regular-equivalences of surface-knot diagrams, where two diagrams representing the same surface-knot are said to be regular-equivalent if they are related by a finite sequence of “branch-free” Roseman moves.

This paper is organized as follows. Section 2 is devoted to reviewing racks, quandles, surface-knots and their diagrams. Our main results, Theorem 3.2 and 3.3, are stated in Section 3. We study rack colorings of surface-knot diagrams with immersed curves in Section 4. Section 5 provides a relationship between a rack coloring with immersed curves and a quandle coloring when surface-knot diagrams which we consider have no branch points. These two sections are devoted to giving the proof of Theorem 3.3, which is proven in the end of Section 5. As mentioned above, Theorem 3.3 implies Theorem 3.2. In Section 6, we discuss a relationship with regular-equivalences of surface-knot diagrams.

2. Definition

2.1. Racks and quandles

For a non-empty set X and a binary operation $*$ on X , we consider the following three conditions.

- (Q1) For any $a \in X$, $a * a = a$.
- (Q2) For any $a \in X$, the map $*a : X \rightarrow X$, defined by $\bullet \mapsto \bullet * a$, is bijective.
- (Q3) For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

These three correspond to the Reidemeister moves of type I, II and III respectively.

A pair $(X, *)$ is called a *quandle* if it satisfies conditions (Q1), (Q2) and (Q3). Quandles are useful for studying oriented knots and also for oriented surface-knots. A pair $(X, *)$ is called a *rack* if it satisfies conditions (Q2) and (Q3). Racks are useful for studying oriented framed knots. We remark that a quandle is a rack by definition and that a rack/quandle $(X, *)$ is often abbreviated to X . Racks and quandles have been studied in, for example, [6, 11, 12].

2.2. Surface-knots and their diagrams

A *surface-knot* (or a Σ^2 -*knot*) is a submanifold of the 4-space \mathbb{R}^4 , homeomorphic to a closed connected oriented surface Σ^2 . We always assume that all surface-knots are oriented in this paper. Two surface-knots are said to be *equivalent* if they can be deformed into each other through an isotopy of \mathbb{R}^4 .

A *diagram* of a surface-knot is its image via a generic projection from \mathbb{R}^4 to \mathbb{R}^3 , equipped with the height information as follows: At a neighborhood of each double point, there are intersecting two disks and one is higher than the other with respect to the 4th coordinate dropped by the projection. Then the height information is indicated by removing the regular neighborhood of the double point in the lower disk along the double point curves. Then a diagram is regarded as a disjoint union of connected compact oriented surfaces, each of which is called a *sheet*. A diagram is basically composed of four kinds of local pictures, each of which is the image of a neighborhood of a typical point — a regular point, a *double point*, an isolated *triple point* or an isolated *branch point*. The latter three are depicted in Figure 1.

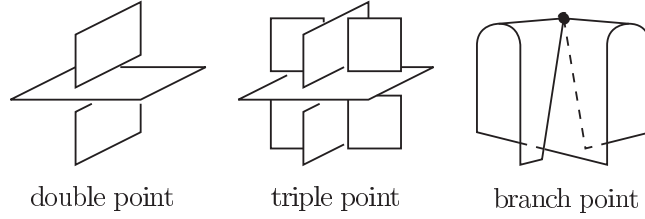


Figure 1: Local picture of the projection image of a surface-knot

Two surface-knot diagrams are said to be *equivalent* if they are related by (ambient isotopies of \mathbb{R}^3 and) a finite sequence of seven Roseman moves, shown in Figure 2, where we omit height information for simplicity. According to Roseman [14], two surface-knots are equivalent if and only if they have equivalent diagrams. We refer to [4] for more details.

3. Rack colorings and main theorem

3.1. Rack colorings

We represent the orientation of a surface-knot diagram by assigning normal directions \vec{n} , depicted by an arrow which looks like the symbol “ \Uparrow ” as in Figure 3, to each sheet of the diagram such that the triple $(\vec{v}_1, \vec{v}_2, \vec{n})$ matches the orientation of \mathbb{R}^3 , where the pair (\vec{v}_1, \vec{v}_2) denote the orientation of the sheet.

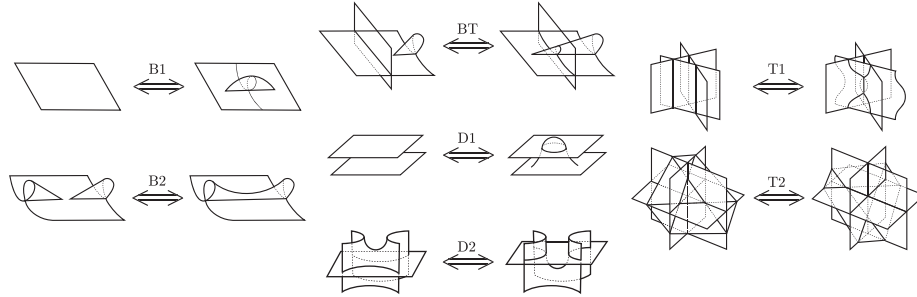


Figure 2: Roseman moves

For a surface-knot diagram D , let $\mathcal{S}(D)$ denote the set of all sheet of D . For a rack R , a map $c : \mathcal{S}(D) \rightarrow R$ is a *rack coloring* if it satisfies the following relation along each double point curve. Let x_j be the over-sheet along a double point curve, and x_i, x_k be under-sheets along the double point curve such that the normal direction of x_j points from x_i to x_k . Then it is required that $c(x_k) = c(x_i) * c(x_j)$ as in Figure 3. Let $\text{Col}_R(D)$ be the set of rack colorings of D by R . We note that when a rack R is finite, $\text{Col}_R(D)$ is also finite. If a rack which we consider is a quandle, then a rack coloring is also called a *quandle coloring*.

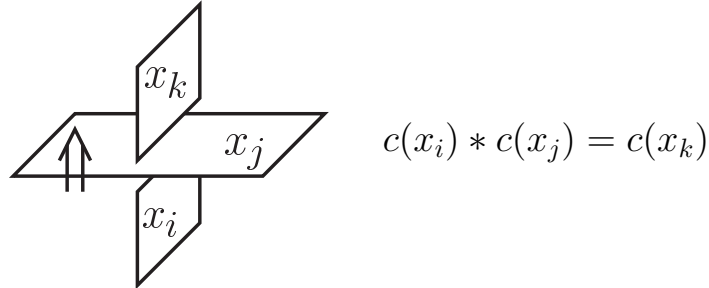


Figure 3: Coloring relation along a double point curve

It is known that a quandle coloring by a quandle Q is an invariant of surface-knots (cf. [2]). Precisely speaking, for two diagrams D_1 and D_2 of a surface-knot, there is a bijection between $\text{Col}_Q(D_1)$ and $\text{Col}_Q(D_2)$. For rack colorings, this is not the case in general. One of the difficulties is the existence of branch points. For example, if a diagram D has branch points and a rack R has no element a such that $a * a = a$, then D admits no rack coloring by R , that is, $\text{Col}_R(D)$ is the emptyset. This is because the equation $a * a = a$ for some $a \in R$ is required as the rack coloring condition along a double point curve one of whose endpoints is a branch point. (We note that a non-quandle connected rack, such as a cyclic rack [6, Example 7], has no element a such that $a * a = a$. Since we do not use the precise definition of the connectedness for racks, we omit the details and give some comments instead. It is known that any rack is decomposed into the connected components each of which is also a rack. It is also known that

any rack coloring for a surface-knot diagram by a rack is essentially the one by a connected component of the rack. Hence the connectedness for racks is suitable in considering rack colorings for surface-knot diagrams.)

3.2. Diagrams without branch points

Even if surface-knot diagrams which we consider have no branch points, we observe, in Example 3.1 below, that rack colorings are not invariants of surface-knots in general. We note that every (oriented) surface-knot has a diagram without branch points (cf. [3]).

Example 3.1. Let D_1 and D_2 be two surface-knot diagrams as in Figure 4. Both represent the same T^2 -knot and do not have branch points. (We note that both represent a trivial T^2 -knot, which bounds a solid torus in \mathbb{R}^4 .) However, for a finite non-quandle connected rack R , the number of $\text{Col}_R(D_1)$ is strictly more than that of $\text{Col}_R(D_2)$, since R has no element a such that $a * a = a$ which is required as a rack coloring condition around the double point curve of D_2 . That is, the number of $\text{Col}_R(D_2)$ is zero, while the number of $\text{Col}_R(D_1)$ is equal to that of R . This is because Roseman moves of type B1 and B2 may change the number of rack colorings by a finite rack in general. We note that these diagrams are appeared in Satoh's paper [16]. We return to this example in Section 6 again.

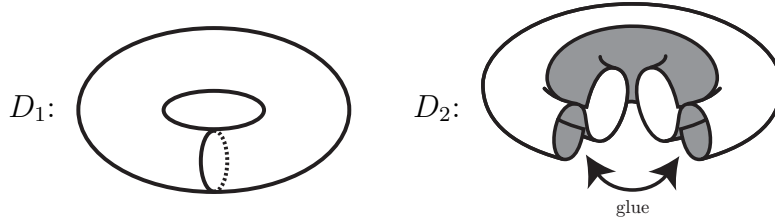


Figure 4: Satoh's T^2 -knot diagrams

3.3. Main theorem

The above situation is totally different for S^2 -knots. In fact, we can prove the following:

Theorem 3.2. *For any two diagrams D_1 and D_2 of an S^2 -knot without branch points, there is a bijection between $\text{Col}_R(D_1)$ and $\text{Col}_R(D_2)$.*

This theorem says that a rack coloring is an invariant of S^2 -knots, even if a sequence of Roseman moves between D_1 and D_2 may involve branch points. For any rack R , we will define the associated quandle, denoted by Q_R , of R in Section 5. Then a rack coloring by R can be interpreted in terms of its associated quandle Q_R as follows:

Theorem 3.3. *For any S^2 -knot diagram D without branch points, there is a bijection between $\text{Col}_R(D)$ and $\text{Col}_{Q_R}(D)$.*

Since quandle colorings are invariants of surface-knots, Theorem 3.3 implies Theorem 3.2. Hence, in what follows, we focus on proving Theorem 3.3. Before that, we will show Theorem 4.4 in Section 4 and Theorem 5.1 in Section 5. Combining these two theorems, we will prove Theorem 3.3 in the end of Section 5.

4. Rack colorings with immersed curves

4.1. The kink map of a rack

For a rack $R = (R, *)$, let $\iota : R \rightarrow R$ be the map, called the (negative) *kink map* [19] of R , characterized by the equation $\iota(a) * a = a$ for any element $a \in R$. The map ι is well-defined by the condition (Q2). It might be better to denote it by a symbol like “ ι_R ”, since this map is uniquely determined by R . However, we denote it by ι for simplicity. We note that (a notion similar to) the map ι has essentially appeared in [13]. See Remark 4.1 below for comments on a hidden diagrammatic meaning of the kink map ι . It is known in [19] that the kink map ι satisfies the following three conditions.

- (K1) The map ι is bijective.
- (K2) For any $a, b \in R$, $\iota(a) * b = \iota(a * b)$.
- (K3) For any $a, b \in R$, $a * \iota(b) = a * b$.

By the condition (K1), there is a unique inverse map, denoted by ι^{-1} in a usual way, of the kink map ι . Then, for any integer n , a symbol ι^n does make sense, that is, the symbol ι^n denote the n -times composition of ι for $n \geq 0$ and the $|n|$ -times composition of ι^{-1} for $n < 0$.

Remark 4.1. The kink map ι of a rack R corresponds to a negative kink of a classical knot diagram in the following sense. The negative kink consists of the two arcs as in Figure 5. If we assign an element $a \in R$ to the arc which comes in, then the arc which goes out receives the element $\iota(a) \in R$ by the rack coloring condition at the crossing, where rack colorings for classical knots are defined in a way similar to that for surface-knots and we omit the details for the definition. Thus we can think of the kink map ι as an algebraic abstraction of the negative kink.

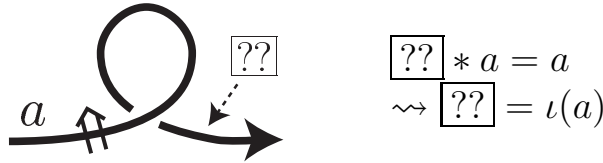


Figure 5: The map kink ι and a negative kink

4.2. Rack colorings with immersed curves

Consider a set L of oriented immersed curves on a surface-knot diagram D . We assume that L intersects itself transversely and each multiple point is a double point. We further assume that L intersects the double point curves of D transversely, and misses triple points and branch points of D . The orientation for each immersed curve of L is represented by normal directions \vec{n} , depicted by an arrow which looks like the symbol “ \uparrow ” as in Figure 6, such that the pair (\vec{v}, \vec{n}) matches the orientation of D , where \vec{v} denote the orientation of the immersed curve.

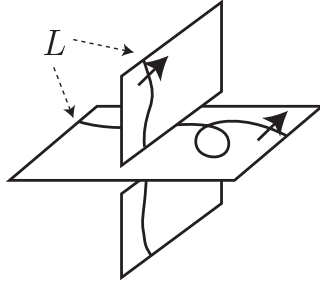


Figure 6: Local picture of oriented immersed curves on a diagram

By cutting each sheet in $\mathcal{S}(D)$ along the curves of L , we obtain a disjoint union of connected compact oriented surfaces, each of which is also called a *sheet* of the pair (D, L) . We note that the local picture of a diagram in Figure 6 consists of seven sheets. We denote by $\mathcal{S}(D, L)$ the set of all sheet of (D, L) . We note that each sheet in $\mathcal{S}(D, L)$ is included in a unique sheet in $\mathcal{S}(D)$ as a subset. For a rack R , a map $c : \mathcal{S}(D, L) \rightarrow R$ is a *rack coloring with immersed curves* if it satisfies the two relation:

- One is the same relation along each double point curve of D as that in (usual) rack colorings, and
- The other is the following relation along each oriented immersed curve of L . Let x_i, x_j be two sheets along an oriented immersed curve such that the normal direction of the curve points from x_i to x_j . Then it is required that $c(x_j) = \iota(c(x_i))$ as in the left of Figure 7.

See Remark 4.2 below for comments on the latter (somewhat artificial) coloring condition. Let $\text{Col}_R(D, L)$ be the set of rack colorings of (D, L) by R . We note that when a rack R is finite, $\text{Col}_R(D, L)$ is also finite. At each double point, say p , of L and double point curves of D , the well-definedness of rack coloring with immersed curves is not so obvious. By using the properties (K2) and (K3) of the kink map ι , we can show the following near the double point p .

- If an arc of L lies on the under-sheets, the condition (K2) ensures the well-definedness at p . The situation is illustrated on the left of Figure 8.

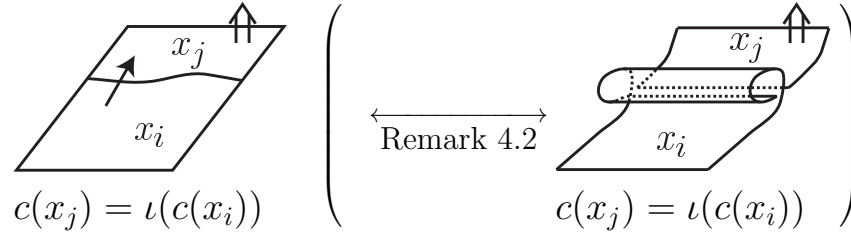


Figure 7: Coloring relation along an oriented immersed curve

- If an arc of L lies on the over-sheet, the condition (K3) ensures the well-definedness at p . The situation is illustrated on the right of Figure 8.

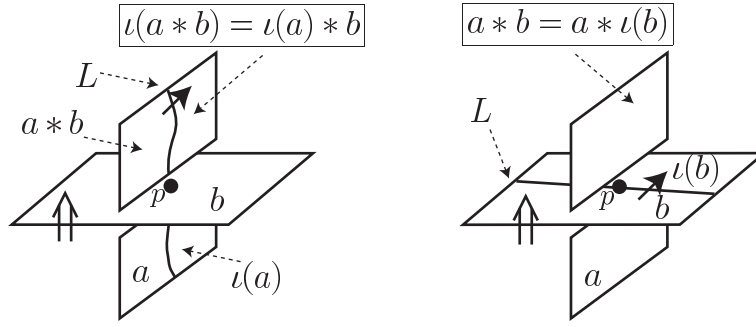


Figure 8: Well-definedness of rack colorings with immersed curves

Remark 4.2. We mention here a hidden meaning of a set L of oriented immersed curves on a surface-knot diagram D . A neighborhood of an arc of L on D is virtually considered as an abstraction of “(kink) $\times[0, 1]$ ” as in the right of Figure 7. If we replace the neighborhood of the arc of L virtually with “(kink) $\times[0, 1]$ ”, then the (usual) rack coloring condition implies our coloring condition along each oriented immersed curve of L .

Proposition 4.3. *For two sets L_1 and L_2 of oriented immersed curves on a surface-knot diagram D , if L_1 is homologous to L_2 , then there is a bijection between $\text{Col}_R(D, L_1)$ and $\text{Col}_R(D, L_2)$.*

Proof of Proposition 4.3. We may assume that L_1 and L_2 , by perturbing L_1 or L_2 if necessary, intersect transversely and each multiple point on the union $L_1 \cup L_2$ is a double point. We denote by $-L_1$ the set L_1 with the opposite orientation. Since L_1 and L_2 are homologous, the union $(-L_1) \cup L_2$ is null-homologous on D .

We choose a sheet, say x_0 , of $(D, (-L_1) \cup L_2)$, and assign an integer to each sheet of $(D, (-L_1) \cup L_2)$ by the following Alexander numbering-like rule:

- We assign 0 to the sheet x_0 .

- For two adjacent sheets x_1 and x_2 along an arc of $(-L_1) \cup L_2$, we suppose that the normal orientation of the arc points from x_1 . Then the integer assigned to x_2 is larger than that to x_1 by 1.
- For two adjacent sheets along a double point curve of D , the integers assigned to them are the same.

See Figure 9. Since $(-L_1) \cup L_2$ is null-homologous on D , this assignment is well-defined.

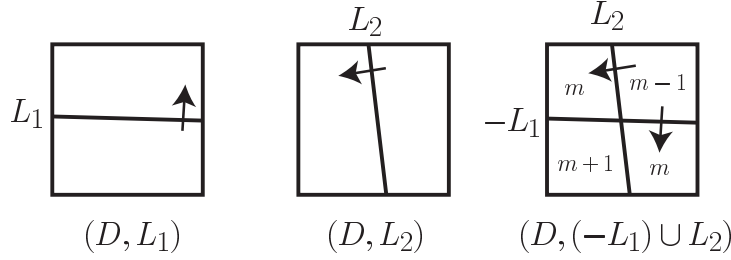


Figure 9: Numbering on $\mathcal{S}(D, (-L_1) \cup L_2)$

Using the above numbering on $\mathcal{S}(D, (-L_1) \cup L_2)$, we construct an explicit bijection $\Phi : \text{Col}_R(D, L_1) \rightarrow \text{Col}_R(D, L_2)$. For a coloring $c \in \text{Col}_R(D, L_1)$, we define $\Phi(c) \in \text{Col}_R(D, L_2)$ as follows: Let x_2 be a sheet in $\mathcal{S}(D, L_2)$. The sheet x_2 might be divided into the further smaller sheets on $(D, (-L_1) \cup L_2)$. (In the case where $L_1 \cap x_2 = \emptyset$, the sheet x_2 is not divided.) For the sheet x_2 , choose one of the small sheets on $(D, (-L_1) \cup L_2)$, and denote it by y . We note that y is included in x_2 as a subset on (D, L_2) . Let x_1 be a unique sheet in $\mathcal{S}(D, -L_1)$ such that x_1 includes y as a subset on $(D, -L_1)$. See Figure 10. Then we define $\Phi(c)(x_2) = \iota^n(c(x_1))$, where n is the numbering on the small sheet y .

The well-definedness of the map Φ is shown as follows: Two adjacent small sheets y and y' , included in x_2 as subsets on $(D, (-L_1) \cup L_2)$, are separated by an arc, say α , of $-L_1$. Suppose that the normal orientation of α points from y . By the above rule, it holds that $n' = n + 1$, where the integer n (resp. n') is the numbering on the small sheet y (resp. y'). Let x_1 and x'_1 be the sheets in $\mathcal{S}(D, -L_1)$ such that x_1 (resp. x'_1) includes y (resp. y') as a subset on $(D, -L_1)$. See Figure 10. By the rack coloring condition for immersed curves, the equation $c(x_1) = \iota(c(x'_1))$ holds. Therefore, we have

$$\iota^n(c(x_1)) = \iota^{n'-1}(c(x_1)) = \iota^{n'-1}(\iota(c(x'_1))) = \iota^{n'}(c(x'_1)).$$

We can also check that the rack coloring conditions along L_2 and the double point curves of D on $\mathcal{S}(D, L_2)$ are satisfied. The inverse map of Φ can be constructed in a way similar to the definition of Φ . (The only difference is to replace n in the definition of Φ with $-n$.)

□

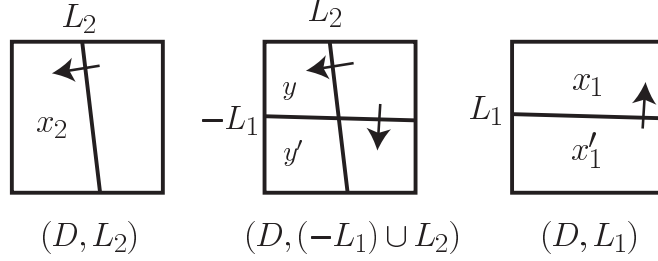


Figure 10: Local picture of the bijection Φ

Since any set of oriented immersed curves on an S^2 -knot diagram is null-homologous, Proposition 4.3 implies:

Theorem 4.4. *For any set L of oriented immersed curves on an S^2 -knot diagram D , there is a bijection between $\text{Col}_R(D, L)$ and $\text{Col}_R(D)$.*

5. Associated quandles

For a rack $R = (R, *)$ and the kink map ι of R , we define a new binary operation $*^\iota$ on the set R by $a *^\iota b := \iota(a) * b$. Then it is known in [1, 19] that the pair $(R, *^\iota)$ becomes a quandle. The quandle $(R, *^\iota)$ is called the *associated quandle* of R and is denoted by Q_R . If a surface-knot diagram has no branch points, then quandle colorings by Q_R can be interpreted in terms of rack colorings with immersed curves by R as follows:

Theorem 5.1. *For any surface-knot diagram D without branch points, there exists a set L of oriented immersed curves on D such that there is a bijection between $\text{Col}_{Q_R}(D)$ and $\text{Col}_R(D, L)$.*

Proof of Theorem 5.1. We take a set L of oriented immersed curves as a copy of the double point curves of D by pushing the double point curves slightly on the under-sheet in the direction opposite to the normal direction of the over-sheet along each double point curve as in the right of Figure 11, where the normal orientation of L matches that of the over-sheet. The set L is well-defined, since the surface-knot diagram D , which we now consider, has no branch points. We note that when D has no triple points (and no branch points), the set L consists of oriented embedded curves.

We construct an explicit bijection $\Psi : \text{Col}_{Q_R}(D) \rightarrow \text{Col}_R(D, L)$. See Figure 11, where we depict a local picture of Ψ along an arc of the double point curves. For a coloring $c \in \text{Col}_{Q_R}(D)$, we define $\Psi(c) \in \text{Col}_R(D, L)$ as follows: Let y be a sheet in $\mathcal{S}(D, L)$. It follows from the construction of L that there exists a unique sheet, say x , in $\mathcal{S}(D)$ such that x includes y as a subset on D . Then we define $\Psi(c)(y) := \iota(c(x))$ if the sheet y lies between an arc of L and the double point curve that is parallel to the arc, and $\Psi(c)(y) := c(x)$ if not.

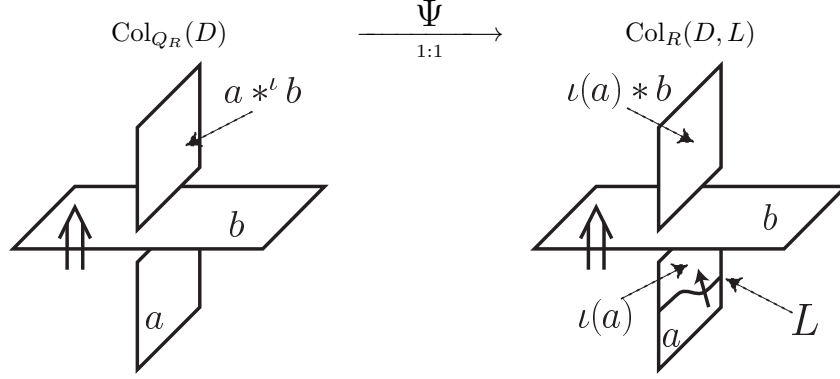


Figure 11: Local picture of the bijection Ψ

This map Ψ is well-defined, since the definition of the binary operation of Q_R requires the equality $a *^\iota b = \iota(a) * b$.

To prove the bijectiveness of Ψ , we construct a map $\Psi' : \text{Col}_R(D, L) \rightarrow \text{Col}_{Q_R}(D)$ and show that Ψ' is the inverse map of Ψ . For a coloring $c' \in \text{Col}_R(D, L)$, we define $\Psi'(c') \in \text{Col}_{Q_R}(D)$ as follows: Let x be a sheet in $\mathcal{S}(D)$. It follows from the construction of L that there exists a unique sheet, say y , in $\mathcal{S}(D, L)$ such that y is included in x as a subset on D and does not lie between an arc of L and the double point curve that is parallel to the arc. Then we define $\Psi'(c')(x) := c'(y)$. Again the definition of the binary operation of Q_R ensures the well-definedness of Ψ' and we can check that Ψ' is the inverse map of Ψ by an explicit calculation. \square

With Theorem 4.4 and Theorem 5.1 in hand, we now prove Theorem 3.3.

Proof of Theorem 3.3. For an S^2 -knot diagram D without branch points, by using Theorem 5.1, there exists a set L of oriented immersed curves on D such that there is a bijection between $\text{Col}_{Q_R}(D)$ and $\text{Col}_R(D, L)$. Then, by using Theorem 4.4, there is a bijection between $\text{Col}_R(D, L)$ and $\text{Col}_R(D)$. By the composition of these two bijections, we have a desired bijection. \square

6. Regular-equivalences and rack colorings

Finally we discuss a relationship with regular-equivalences of surface-knot diagrams. As we mentioned in Subsection 3.2, any (oriented) surface-knot has a diagram without branch points. In [16], such a “branch-free” diagram is called a *regular* diagram, and two regular diagrams are said to be *regular-equivalent* if they are related by a finite sequence of “branch-free” Roseman moves, that is, the moves of type $D1$, $D2$, $T1$ and $T2$ in Figure 2. (For surface-knot diagrams, which may have branch points, we can also define the notion of “regular-equivalence” in a similar way. That is, two surface-knot diagrams are said to

be *regular-equivalent* if they are related by a finite sequence of “branch-free” Roseman moves.) We can easily check that there is a one-to-one correspondence between rack colorings before and after each “branch-free” Roseman move. (We note that there is also a one-to-one correspondence before and after the Roseman move of type *BT*.) Then we have the following:

Theorem 6.1. *For two diagrams D_1 and D_2 of a surface-knot, if they are regular-equivalent, then there is a bijection between $\text{Col}_R(D_1)$ and $\text{Col}_R(D_2)$.*

This theorem says that rack colorings are invariants of regular-equivalence classes of diagrams of a surface-knot. Using a contraposition of Theorem 6.1, we can reprove that Satoh’s examples are not regular-equivalent, that is, Example 3.1 implies the following:

Corollary 6.2. *([16, Theorem 3]) There exist two regular surface-knot diagrams which are equivalent but not regular-equivalent.*

Theorem 6.1 cannot tell us anything about regular-equivalence classes of diagrams of an S^2 -knot. Precisely speaking, for two diagrams D_1 and D_2 of an S^2 -knot, it follows from Theorem 3.2 that even if they are not regular-equivalent, there is a bijection between the two sets, $\text{Col}_R(D_1)$ and $\text{Col}_R(D_2)$, of rack colorings by a rack R . However, Takase and the second author [17] proved that, for any regular S^2 -knot diagram D , there exists a regular diagram D' such that D and D' represent the same S^2 -knot but they are not regular-equivalent. They used immersion theory to prove such a theorem instead of rack theory.

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